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Supertransparent potentials for the Dirac equation

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Abstract. An extension of the classical Darboux transformations is applied to the one-dimensional Dirac equation in order to construct von Neumann–Wigner potentials allowing embedded eigenvalues. These potentials lead to a novel type of scattering problem with a trivial S-matrix composed of vanishing reflection coefficients and a trivial transmission coefficient. Related topics like the underlying symmetry of the Dirac equation and the connection with positon solutions of nonlinear evolution equations are discussed.

1. *Introduction*

The one-dimensional Dirac equation with a Lorentz scalar (and/or vector) potential has recently received much attention. These studies cover such diverse areas as the traces of supersymmetry in one dimension [1], one-dimensional nuclear models (see, e.g., the references cited in [2]), technical questions on topics such as the existence of bound states or boundary conditions for singular potentials [3, 4], the remnants of the well known chiral symmetry of the 3D Dirac equation in the massless limit [5], the construction of transparent potentials and the spectral properties resulting from the inherent supersymmetric structure [6] and relativistic tunnelling problems [7]. An inspection of the literature shows that most exactly solvable models for the one-dimensional Dirac equation employ the close relation between this equation and a pair of Schrödinger equations; the solution of these Schrödinger equations, in turn, is then based on well established algebraic techniques.

It is shown in the present paper that these strategies can be considerably simplified by applying Darboux transformations (DT) directly to the Dirac equation. This approach requires nothing but an adaption of DT, which are well known from the theories of nonlinear waves to relativistic quantum mechanics. To be specific, the construction of long-ranged oscillatory potentials admitting bound states embedded in the continuum (BSEIC) will be presented below. These potentials turn out to be supertransparent; what is defined here as having a trivial S-matrix: they lead to vanishing (left and right) reflection coefficients while the transmission coefficients are identically equal to one (i.e. $T = 1$). This behaviour differs drastically from the solitonic potentials which have the same reflection coefficients but the transmission coefficient of which is a complex number of modulus one [8]. Thus the supertransparent potentials constructed here present another aspect of the problem of relativistic tunnelling phenomena (cf [7]), considerably increase the set of reflectionless potentials (cf [6]) and provide explicit examples of the symmetry of the one-dimensional Dirac equation discussed in [5].

The paper is organized as follows: in the next section, the notation used below is set up and the DT underlying the present work are introduced. In section three, BSEIC-bearing potentials for the one-dimensional Dirac equation are constructed and their

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supertransparency is shown. The following sections contain remarks on related topics comprising: (i) the (super-)symmetry of the Dirac equation and the Miura transformation; (ii) the relation to positon solutions of nonlinear evolution equations. The paper is concluded by a short summary.

2. Darboux transformations of the Dirac equation

The one-dimensional massless Dirac equation is taken in the form

$$(-\mathrm{i}\sigma_1\partial_x + \sigma_3 v)\phi = \lambda\phi \tag{1}$$

where appropriate units have been chosen, the σ_i $(i = 1, 2, 3)$ are Pauli spin-matrices and ϕ is a two-component spinor $\phi = (\psi, \varphi)^T$. It is, however, more convenient to work in a representation where the unitary involution $\tau = \mathrm{i}\sigma_1\sigma_3$ is diagonal. Thus, (1) is transformed to

$$(-\mathrm{i}\sigma_2\partial_x + \sigma_1 v)\phi = \lambda\phi \tag{2}$$

which will be the basis for the present discussion.

The following facts and properties around equation (2) will be used from here on.

(i) The DT constructed and employed here focus on (2), but also hold, of course, upon simple modifications, for other forms of the Dirac equation. A survey of various representations and the corresponding transformations can be found, for example, in [8].

(ii) The scattering discussed below does not require the potential $v(x)$ to be continuous. ([9] contains a study of scattering for singular potentials in the one-dimensional Dirac equation developed in the frame of nonlinear evolution equations.)

(iii) The eigenvalues occur in pairs $\pm\lambda$ (see e.g. [3]). This corresponds to the fact that one can form two scattering solutions with the same energy moving to the right and to the left, respectively. The spectrum in the presence of a mass-term and the peculiarities connected with potentials having non-trivial spatial asymptotics is discussed in [3, 4, 8].

(iv) The Dirac equation (2) is related via supersymmetric quantum mechanics to a pair of Schrödinger equations reading

$$H_j\psi = (-\partial_{xx} + V_j)\psi = \lambda\psi \tag{3}$$

where the potentials V_j in (3) and v in (2) satisfy $(j = 1, 2)$

$$V_j = v^2 + (-1)^j v_x. \tag{4}$$

Therefore the concepts well known from Schrödinger-theory have natural extensions to the relativistic case (cf [8, 10]). The Dirac Jost-solutions needed below, for instance, can be derived from Schrödinger Jost-solutions. Restricting for definiteness to the positive eigenvalue, they can be written as

$$F^{(\mathrm{l})} = \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \mathrm{e}^{-\mathrm{i}kx} \qquad F^{(\mathrm{r})} = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \mathrm{e}^{+\mathrm{i}kx} \tag{5}$$

$$G^{(\mathrm{l})} = \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} \mathrm{e}^{-\mathrm{i}kx} \qquad G^{(\mathrm{r})} = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \mathrm{e}^{+\mathrm{i}kx} \tag{6}$$

where (l), (r) stand for solutions going to the left and to the right. (The two sets F, G correspond to the degeneracy.) The transmission coefficient T and the reflection coefficient to the right R^{r} are defined via

$$TF^{(\mathrm{l})} = G^{(\mathrm{l})} + R^{\mathrm{r}}F^{(\mathrm{r})}. \tag{7}$$

A similar definition holds for the reflection coefficient to the left. These coefficients constitute the S-matrix via

$$S = \begin{pmatrix} T & R^{\mathrm{r}} \\ R^{\mathrm{l}} & T \end{pmatrix}. \tag{8}$$

(v) The supertransparency phenomenon is now defined by the property that the S-matrix becomes the unit matrix, i.e. $T = 1$, $R^{\mathrm{r}} = R^{\mathrm{l}} = 0$.

In order to derive DT for the Dirac equation (2), it is convenient to rewrite it in the form

$$\phi_x = U\phi \tag{9}$$

with

$$U = \begin{pmatrix} -v & -\lambda \\ \lambda & v \end{pmatrix}. \tag{10}$$

A DT of (9) can now be defined as a linear transformation of the eigenfunctions

$$\tilde{\phi} = P\phi \tag{11}$$

such that (9) is invariant under (11) upon replacing the potential v by the transformed potential $\tilde{v}$. A DT of (2), written in the form (9), (10), is derived from a convenient formal generalization of (9), (10) to (cf [11])

$$\Phi_x = \sum_{j=0}^{1} U_j \Phi \Lambda^j \tag{12}$$

where Φ is a 2×2 matrix-valued function whose columns are two solutions of the original equation (9), (10) with eigenvalues $\pm\lambda_1$, Λ is a diagonal matrix with $\Lambda = (\lambda_1, -\lambda_1)$ and $U_{0,1}$ are 2×2 matrices. These are determined by: (i) noting that the formal equation (12) contains, by construction, two copies of (9), (10) for $\pm\lambda_1$; (ii) postulating equivalence of (9), (10) and (12); (iii) keeping in mind that the elements of U_1 involve the eigenvalues $\pm\lambda_1$ while those of U_0 involve the potential v, and (iv) comparing both equations.

It turns out that (12) is invariant under the DT $\Phi \to \Phi[1]$, with

$$\Phi[1] = \Phi\Lambda - \sigma\Phi \qquad \sigma = \Phi_1\Lambda_1\Phi_1^{-1}. \tag{13}$$

Here, Φ_1 is a fixed solution of (12) with fixed eigenvalue $\lambda_1 := \lambda_{11}$, $\Lambda_1 = (\lambda_{11}, -\lambda_{11})$. The coefficients of (12) transform according to

$$U_1[1] = U_1 \qquad U_0[1] = U_0 + [U_1, \sigma] \tag{14}$$

where $[A, B]$ denotes the usual commutator $AB - BA$.

Thus, one obtains DT of the Dirac equation (2) (and other forms) by proceeding along the following lines:

(i) rewrite the Dirac equation in the form (9), (10);

(ii) determine the matrices U_0, U_1 introduced in (12);

(iii) determine a matrix solution Φ_1 for a pair of eigenvalues $(\pm\lambda_{11})$ and compute σ; and

(iv) obtain the new potential $v[1]$ and the corresponding eigenfunction from the DT (13) and (14).

Denoting the special solution Φ_1 of (13) by

$$\Phi_1 = \begin{pmatrix} \psi_1 & \psi_2 \\ \varphi_1 & \varphi_2 \end{pmatrix} \tag{15}$$

one obtains, as a result of a one-step DT for the Dirac equation (2),

$$v[1] = \frac{\Delta q[1] + \Delta r[1]}{\Delta[1]} \qquad \phi[1] = \frac{1}{\Delta[1]}\begin{pmatrix} -\Delta\psi[1] \\ \Delta\varphi[1] \end{pmatrix} \tag{16}$$

with

$$\begin{aligned} \Delta[1] &= \det\begin{pmatrix} \psi_1 & \psi_2 \\ \varphi_1 & \varphi_2 \end{pmatrix} \\ \Delta q[1] &= \det\begin{pmatrix} \psi_1 & \psi_2 \\ \lambda_1\psi_1 & -\lambda_1\psi_2 \end{pmatrix} \qquad \Delta r[1] = \det\begin{pmatrix} \varphi_2 & \varphi_1 \\ -\lambda_1\varphi_2 & \lambda_1\varphi_1 \end{pmatrix}. \end{aligned} \tag{17}$$

The transformed eigenfunction is given by

$$\Delta\psi[1] = \det\begin{pmatrix} \psi_1 & \psi_2 & \psi \\ \lambda_1\psi_1 & -\lambda_1\psi_2 & \lambda\psi \\ \varphi_1 & \varphi_2 & \varphi \end{pmatrix} \qquad \Delta\varphi[1] = \det\begin{pmatrix} \varphi_1 & \varphi_2 & \varphi \\ \lambda_1\varphi_1 & -\lambda_1\varphi_2 & \lambda\varphi \\ \psi_1 & \psi_2 & \psi \end{pmatrix}. \tag{18}$$

Choosing, for example, as starting functions for $v_0 = 0$, the solutions

$$\psi_1 = e^{i\lambda_1 x} + e^{-i\lambda_1 x} \qquad \varphi_1 = -ie^{i\lambda_1 x} + ie^{-i\lambda_1 x} \tag{19}$$

and ψ_2, φ_2 correspondingly with $\lambda_1 \to -\lambda_1$, leads to the new potential

$$v[1] = \frac{8\lambda_1}{2i(e^{2i\lambda_1 x} - e^{-2i\lambda_1 x})} \tag{20}$$

which gives, for $\lambda_1 = i\mu_1$ pure imaginary, the result

$$v[1] = \frac{-2\mu_1}{\sinh(2\mu_1 x)} \tag{21}$$

and for λ_1 real, one obtains the singular periodic potential

$$v[1] = \frac{-2\lambda_1}{\sin(2\lambda_1 x)}. \tag{22}$$

The corresponding eigenfunction is easy to calculate by inserting the general solution $(\psi, \varphi)^T$ of (9) for $v = 0$ in the determinants listed above.

An iteration of this ansatz is achieved either by direct computation or by the observation that the DT (13), (14) for the Dirac equation (2) as expressed in (15)–(18) is a simple modification of the DT for the nonlinear Schrödinger equation formulated in [13], when time t is put equal to zero. Equations (13)–(18) correspond to equations (8) in [13]. The obvious generalization to an n-fold DT follows from the results given in [13] as

$$\Delta\psi[n] = \frac{d_1(2n+1)}{d(2n)} \qquad \Delta r[n] = \frac{d_2(2n)}{d(2n)} \tag{23}$$

$$d(2n) = \det(A_{ik}),\, A_{ik} = \begin{cases} \lambda^{i-1}\psi_k & i = 1,\ldots,n \\ \lambda_k^{i-1-n}\varphi_k & i = n+1,\ldots,2n \end{cases} \tag{24}$$

$$d_2(2n) = \det(C_{ik}),\, C_{ik} = \begin{cases} \lambda_k^{i-1}\varphi_k & i = 1,\ldots,n+1 \\ \lambda_k^{i-2-n}\psi_k & i = n+2,\ldots,2n \\ k = 1,2,\ldots,2n \end{cases} \tag{25}$$

$$d_1(2n+1) = \det(B_{ik}),\, B_{ik} = \begin{cases} \lambda_k^{i-1}\psi_k & i = 1,\ldots,n+1 \\ \lambda_k^{i-2-n}\varphi_k & i = n+2,\ldots,2n+1 \\ k = 1,2,\ldots,2n+1 & \psi_{2n+1} := \psi \qquad \varphi_{2n+1} = \varphi. \end{cases} \tag{26}$$

The expressions for $-\Delta\varphi[n]$, $\Delta q[n]$ require the interchange of φ and ψ in the numerators of these expressions. The supertransparent potentials constructed in the next section result in this frame from a two-fold DT for a pair of eigenvalues $(\lambda_1, -\lambda_1)$ and $(\lambda_2, -\lambda_2)$, and subsequently computing the limit $\lambda_2 \to \lambda_1$ in this DT.

The DT of the Dirac equation as discussed above are just one possibility offering certain computational advantages; for the sake of completeness some related approaches shall be listed here to conclude this section: probably the most closely related DT of Dirac-like systems, based on an ansatz for the matrix P defined in (11), have been formulated in [14] in the context of nonlinear evolution equations. A modest variation of this strategy starting from an appropriate ansatz for the Jost solutions can be extracted from the references cited in [15]. Very general DT, applicable to $n \times n$ first-order systems, can be found in [16, 17]. An independent development of DT, directly related to the Dirac equation (2), has been presented in [18], where the method of intertwining has been transferred from the theory of solvable Schrödinger equations to the Dirac equation.

3. Supertransparent potentials of the Dirac equation

Supertransparent potentials of the Schrödinger equation are von Neumann–Wigner potentials, defined as

$$V(r) = \frac{a\sin(kr+\eta)}{r} + V_s + V_l \tag{27}$$

which: (i) have a BSEIC iff $|a| > |k|$; and (ii) lead to phaseless scattering over the half-line (a trivial S-matrix over the full line). In (27), V_s and V_l denote short-ranged and long-ranged non-oscillatory components of the potential $V(r)$, respectively. The $1/r$ decay of the oscillatory component has been chosen for convenience in the calculation; in principle, it could be replaced by a decay $\sim 1/r^\alpha$ with $0 < \alpha \leqslant 1$. (A construction of supertransparent potentials for the Schrödinger equation with an arbitrary number of BSEIC via extended DT has been presented in [19]). The relation between the Dirac and Schrödinger equations as expressed in (2)–(4) indicates already that the concept of supertransparent potentials does carry over to the Dirac equation.

The study of local BSEIC-bearing potentials for the Schrödinger equation started in 1929 with [20] and many explicit examples not discussing supertransparency are known; the knowledge about such potentials for the Dirac equation, by contrast, is comparatively poor. (The 'opposite' question, i.e. the search for conditions assuring the non-existence of embedded eigenvalues for the Dirac equation, is under active study, as the references cited in [21] prove.) Explicit construction of BSEIC-bearing potentials for the one-dimensional Dirac equation are extremely rare; early studies were devoted to conditions for the existence of embedded eigenvalues [22] and the corresponding asymptotics [23]. An interesting approach to this field based on asymptotic integration and singularities of the m-function can be found in [24], where many interesting references can also be found. (In [24], the author analyses very general classes of von Neumann–Wigner potentials defined only by their asymptotic properties and discusses the interplay between BSEIC and resonances.)

The simplest supertransparent potential for the Dirac equation one can obtain in the framework set up here results from putting $n = 2$ in the general n-fold DT discussed in the previous section, inserting the functions ($v_0 = 0$)

$$\psi_j = \mathrm{e}^{\mathrm{i}\theta_j} + \mathrm{e}^{-\mathrm{i}\theta_j} \qquad \varphi_j = -\mathrm{i}\mathrm{e}^{\mathrm{i}\theta_j} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\theta_j} \tag{28}$$

with

$$\begin{aligned} &\theta_j = \lambda_j(x + x_1(\lambda_j)) \qquad j = 1, \ldots, 4 \\ &\mathrm{Im}\, x_1 = \mathrm{Im}\, \lambda_j = 0 \qquad \lambda_2 = -\lambda_1 \qquad \lambda_4 = -\lambda_3 \end{aligned} \tag{29}$$

in (23)–(26) and calculating the potential and eigenfunction(s) resulting from the limit $\lambda_3 \to \lambda_1$. (This limit is well defined since the solutions $(\psi_3, \varphi_3)^T$ and $(\psi_4, \varphi_4)^T$ are analytical functions of x and the spectral parameter.) Inserting the Taylor expansion of $(\psi_3, \varphi_3)((\psi_4, \varphi_4))$ around the point $\lambda_3 = \lambda_1(-\lambda_3 = -\lambda_1)$ into the determinant representation of DT given above, one obtains the potential

$$v(x) = \frac{4\lambda_1(\sin 2\theta - 2\lambda_1\gamma\cos 2\theta)}{\sin^2 2\theta - 4\lambda_1^2\gamma^2} \tag{30}$$

where

$$\theta = \lambda_1(x + x_1(\lambda_1)) \qquad \gamma = \theta_{\lambda_1} = x + x_2 \qquad x_2 = x_1 + \lambda_1\partial_{\lambda_1}x_1(\lambda_1). \tag{31}$$

This potential has obviously two first-order poles determined by

$$\sin^2 2\theta - 4\lambda_1^2\gamma^2 = 0 \tag{32}$$

whose exact location can be fine-tuned by the choice of the parameters λ_1, x_1 and x_2. For $x \to \pm\infty$, one obtains the asymptotic estimate

$$v(x) = \frac{2}{\lambda_1 x}\cos(2\theta)\left[1 + \mathrm{O}\left(\frac{1}{x}\right)\right] \tag{33}$$

i.e. potential (30)—depicted in figure 1—is a von Neumann–Wigner potential, as defined in (27). In order to prove the supertransparency of (30), the functions

$$\Phi[2] = \frac{1}{\Delta[2]}\begin{pmatrix} -\Delta\psi[2] \\ \Delta\varphi[2] \end{pmatrix} \tag{34}$$

have to be calculated, again using the Taylor expansion for the limit $\pm\lambda_3 \to \pm\lambda_1$. (The details of this, in principle, straightforward calculation are omitted. It should be noted, however, that the proof of supertransparency is based on the possibility of introducing appropriate scattering data for the potentials considered here, which are—when defined over the full line—in general, singular; scattering data for singular potentials are not uniquely defined. The choice suggested above was made with the conditions on the existence of embedded eigenvalues discussed in [22–24] in mind.)

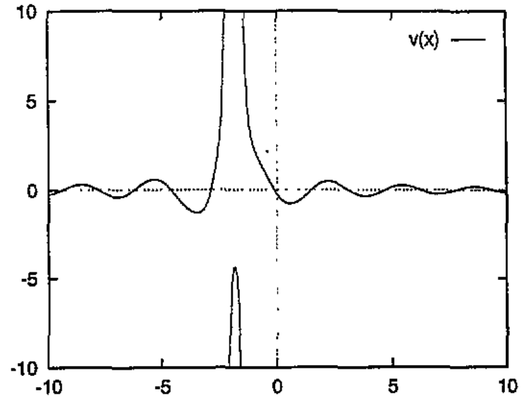

Figure 1. The figure shows potential (30) for $\lambda_1 = 1$, $x_1 = \pi/4$, $x_2 = 2$. The two poles and the regular behaviour for $x > 0$, i.e. in the 'physical' region, are clearly visible.

In the next step, the asymptotics for $x \to \pm\infty$ of the eigenfunction obtained in the aforementioned limit have to be evaluated. From this asymptotic form, one obtains the Dirac Jost-solution, defined in (5), (6), explicitly by comparison. These Jost-solutions, in turn, allow one to determine the transmission and reflection coefficients by inserting them into the corresponding definitions (i.e. (7)). Following this strategy, one obtains, in the present case, indeed, the result that the potential $v(x)$ in (30) leads to vanishing reflection coefficients ($R^{\mathrm{r}} = R^{\mathrm{l}} = 0$), while the transmission coefficients are identically equal to one (i.e. $T = 1$). This result concludes the proof of supertransparency. An extension to a supertransparent potential with two BSEIC is, in principle, straightforward; the only technical difficulty is the evaluation of 8×8 determinants for the potential and 9×9 determinants for the eigenfunction. (The strategy follows the same lines as before: put $n = 4$ in the DT, introduce the solutions (ψ_j, φ_j) of the Dirac equation for vanishing background potential $v_0 = 0$, and calculate the DT over the limit $\lambda_3 \to \lambda_1$, $\lambda_6 \to \lambda_4$ while using the fact that the eigenvalues actually occur in pairs $\pm\lambda_k$, $k = 1, 2, 3$.) The general result is not displayed here due to its length; an explicit form for $\lambda_1 = 1$, $\lambda_2 = 2$ with all phases put identically to zero reads

$$v(x) \equiv \frac{f_1(x)}{f_2(x)} \tag{35}$$

with

$$\begin{aligned} f_1(x) = 12(&108x^3\cos^4 x - 41\sin 2x + 45x^2\sin 2x + 49\sin x\cos^7 x - 87\sin x\cos^5 x \\ &- 22\sin x\cos^9 x + 142\sin x\cos^3 x - 171x^2\sin x\cos^3 x \\ &+ 180\cos^2 x + 12x\cos^{10} x - 18x\cos^8 x - 219x\cos^4 x + 63x\cos^6 x - 18 \\ &- 81x^3\cos^2 x) \end{aligned} \tag{36}$$

$$f_2(x) = 81x^4 - 180x^2 + 64 + 180\cos^2 x - 1167\cos^4 x + 1534\cos^6 x - 603\cos^8 x - 12\cos^{10} x \\ - 360x^2\cos^6 x + 162x^2\cos^2 x + 54x^2\cos^4 x + 324x^2\cos^8 x + 4\cos^{12} x \\ - 864x\sin x\cos^7 x + 1080x\sin x\cos^5 x - 216x\sin x\cos^3 x. \qquad (37)$$

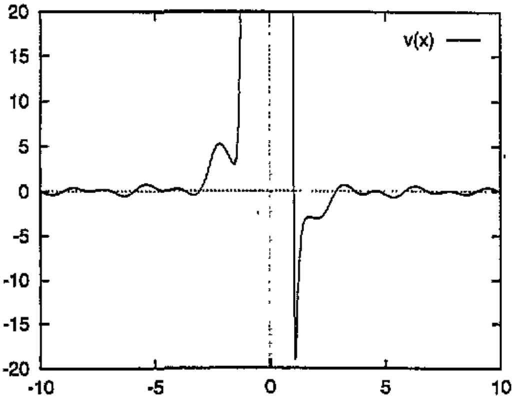

Figure 2. By appropriate choice of the parameters, the singularities of the 'two-BSEIC potential can obviously be located in vicinity of the origin. (The location of the singularities is very sensitive to the parameters.)

Figure 2 demonstrates the remarkable symmetry of this potential with respect to the y-axis (apart from the complicated structure in the vicinity of the origin).

Successive applications of DT following this strategy result in potentials with n BSEIC. (The only technical difficulty is the increasing size of the determinants involved in the calculation.) As expected, these potentials are also supertransparent. More general supertransparent potentials can be obtained by allowing higher-order degeneracies in the eigenvalue, i.e. taking n distinct eigenvalues λ_j and considering the solutions resulting from the limits $\lambda_k \to \lambda_1$, $k = 2, \ldots, n$. This ansatz requires the evaluation of $2n \times 2n$ determinants, now containing partial derivatives up to order n in the spectral parameter. It is somewhat surprising that the resulting potentials can be divided in classes according to n even or n odd; it turns out that only potentials with n odd are supertransparent. In this context, the potential $v(x) = -2\lambda_1/\sin(2\lambda_1 x)$ of equation (22) is 'of order zero' since no derivative occurs.

It should be noted that the formulation of DT developed in [14], when applied to the present problem, leads to identical results. The only difference results from the fact that the calculation of the eigenfunctions is based on a different strategy and is therefore slightly more involved than in the present approach.

In the case of n simple BSEIC, as well as BSEIC obtained from higher-order degeneracies, the supertransparency of these potentials can be shown in two ways. The first possibility is direct calculation in complete analogy to the proof of supertransparency sketched above for potential (30) with one BSEIC. The logically straightforward, but technically involved, calculation can be modelled after the stragegy used in [25] to prove the supertransparency of positons of the sine–Gordon equation. (Positons will be defined below). The second possibility is provided by the supersymmetry of the Dirac equation (2) manifesting itself in the connection between (2) and equations (3) and (4): loosely speaking, the S-matrix of the Dirac equation (2) follows from the S-matrices of the Schrödinger equations (3) and (4) (cf [8]). Thus, the proof of supertransparency boils down to the proofs for the corresponding potentials of the Schrödinger equation. (This correspondence between supertransparent potentials in the relativistic and non-relativistic cases is demonstrated in the next section for

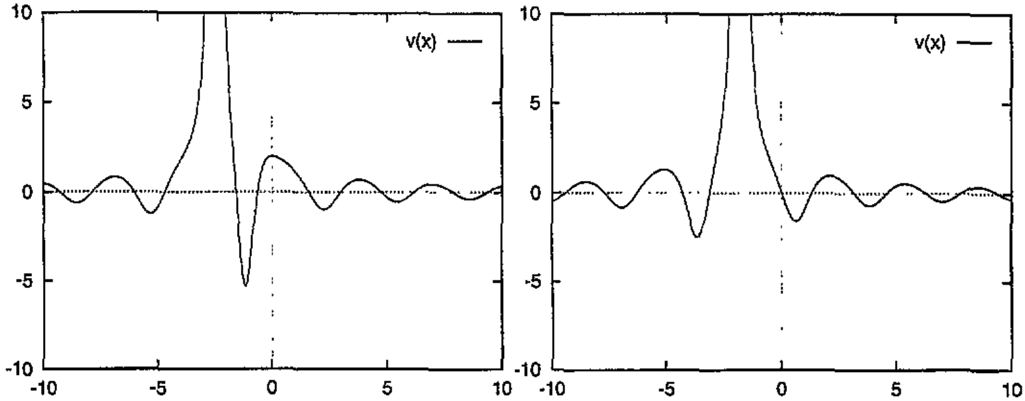

Figure 3. The result of Miura transformation (4), with a '+' sign for potential (30), using the parameters $\lambda_1 = 1$, $x_1 = \pi/2$, $x_2 = 2$.

Figure 4. The second Miura transformation with the same parameters as in figure 3. The resulting figure agrees with figure 1 of [19] as explained in the text.

potential (30).) The non-relativistic case, in turn, allows a variety of strategies for proving the supertransparency of these potentials, summarized in [19]. Probably the most elegant proof based on S-matrix theory can be found in [26]. (The possibility of S-matrix-based proofs is indicated in the conclusion, below). A proof of supertransparency for the non-relativistic case, based on direct calculation, can be deduced from the results presented in [27].

4. Supersymmetry and Miura transformations

It has been observed by many authors that the Dirac equation (2) and the Schrödinger equations (3) and (4) are related via supersymmetric quantum mechanics (see, e.g., [1, 6, 8, 19]). This formalism allows one to deduce the spectral properties of (2) from those of equations (3) and (4); the solution of these Schrödinger equations, in turn, can be based on well established algebraic techniques. Relation (4) between the potentials, here cited in the frame of supersymmetric quantum mechanics, is known in the theory of *nonlinear evolution equations as the Miura transformation, relating solutions of the modified* Korteweg de Vries (KdV) equation to those of the KdV equation [8]. It is now interesting to note that the Miura transformation (4) allows one to derive BSEIC-bearing supertransparent potentials for the Schrödinger equation starting from the results for the Dirac equation. In the theory of nonlinear waves, by contrast, the properties of solitons are not automatically preserved but require an additional coordinate transformation. Taking the supertransparent potential (30) with one BSEIC as an explicit example, this statement can be trivially verified. One obtains the following potential corresponding to the 'one-positon potential', discussed in detail in [27]:

$$V(x) = \frac{32\lambda_1^2 \sin\theta(\sin\theta - 2\lambda_1\gamma\cos\theta)}{(\sin 2\theta - 2\lambda_1\gamma)^2} \tag{38}$$

where θ and γ have been defined in (31). Figures 3 and 4 show the results of Miura transformation (4) with a '+' and '−' sign, respectively; it is trivial to verify that these potentials correspond to: (i) inserting $x_1 = \pi/2$, $x_2 = 2$; and (ii) inserting $x_1 = \pi$, $x_2 = 2$ in the one-positon potential above. Thus, the Miura transformation does indeed preserve the properties of supertransparent potentials.

5. Symmetries of the Dirac equation

The properties of supertransparent potentials are not only preserved in a transition from the Dirac equation (2) to the Schrödinger equation(s), but also in the opposite direction. When constructing supertransparent potentials for the Schrödinger equation, with n BSEIC via DT, one starts, for a vanishing background, from the ansatz [19]

$$\psi \to \psi[n] = \frac{W_2}{W_1} \qquad V_0 \to V[n] = V_0 - 2(\log(W_1))_{xx} \tag{39}$$

where the Wronski determinants W_k are constructed according to

$$W_1 = W(\phi_1, \phi_{1\kappa_1}, \phi_2, \phi_{2\kappa_2}, \ldots, \phi_n, \phi_{n\kappa_n}) \tag{40}$$

$$W_2 = W(\phi_1, \phi_{1\kappa_1}, \phi_2, \phi_{2\kappa_2}, \ldots, \phi_n, \phi_{n\kappa_n}, \psi) \tag{41}$$

with

$$\phi_i = \sin\theta_i \qquad \theta_i = \kappa_i(x + x_1(\kappa_i)) \qquad \psi = \mathrm{e}^{ikx}. \tag{42}$$

Eigenfunction $\psi[n]$ allows one to determine supertransparent potentials for the Dirac equation (2) via

$$v[n] = \pm\partial_x(\log\psi[n](x))|_{k=0}. \tag{43}$$

For $n = 1$, only κ_1 has to be inserted in the Wronskians given above; it is a short calculation to show that one indeed obtains the supertransparent potential (30) by this strategy. (More details can be found in [28].)

The ambiguity in the $\pm$ sign in the log derivative leads immediately to the question of whether a symmetry exists which is responsible for this degree of freedom. The answer to this question has been given in [3, 5]. It is a remnant of the chiral symmetry of the Dirac equation in the massless limit when restricted to one dimension, as is done here; a particle moving in this potential is 'blind' to the sign of the potential (which can be verified by employing the '$-$' sign in the calculation).

6. Positons and supertransparent potentials

Dirac equation (2) appears in the linear representation of the modified KdV equation as

$$v_t = 6v^2 v_x - v_{xxx} \tag{44}$$

while a Schrödinger equation constitutes one part of the Lax pair of the KdV equation. The supertransparent potentials considered here lead, in this context, to a new type of nonlinear waves called positons (cf [28]); all DT derived above remain valid when the argument of the functions is augmented by the corresponding time dependence.

Positons of the modified KdV equation are weakly localized long-ranged singular solutions. The supertransparency manifests itself in the fact that they are completely transparent in interactions with other nonlinear waves: the positon–positon collision is a phaseless event, where the waves involved do not experience any phase shift; the soliton–positon collision leads to well defined phase shifts for the positon which is itself completely transparent to the soliton (see [27] for the KdV equation).

7. Conclusions

Supertransparent real potentials of the one-dimensional Dirac equation have been constructed and discussed. The technical tool needed for this purpose is a generalization and adaption of DT constructed some time ago in the theory of nonlinear waves. In order to obtain the so-called optical potentials for the Dirac equation including imaginary components, one has to take in the frame used here the form of the Dirac equation discussed in [3]. When inverting the sign of the potential considered there and restricting the discussion to complex eigenvalues, one deals in fact with the Dirac equation used in the linear representation of the sine–Gordon equation. The DT leading, for example, to positon–soliton solutions of this nonlinear equation are, in the present context, optical potentials upon setting the time t equal to zero. A discussion of the DT needed for this purpose can be found in [29] and the references cited therein.

The supertransparency phenomenon has a natural explanation in terms of S-matrix theory when looking at the construction of these potentials: two continuum solutions of the free Dirac equation with different eigenvalues λ_1, λ_2 are plugged into the Darboux formalism and the limit $\lambda_2 \to \lambda_1$ is considered. This enforced degeneracy, in turn, leads to new potentials with a bound state in the continuum and trivial scattering, i.e. to a result predicted, in the context of the Schrödinger equation, as a consequence of 'accidental degeneracy of resonances' [30]. The corresponding results for the Schrödinger equation follow directly from the unitarity of the S-matrix [19, 30]; the present work demonstrates that the discussion in [30] can be extended to the relativistic case. Furthermore, the BSEIC-bearing potentials, discussed here, guarantee a full transmission without imposing certain conditions on the parameters of the potential; the 'destructive interference of multiple reflected waves' cited in [7] in the context of discontinuous potentials is achieved in the present continuous case by the oscillating and slowly decaying character of the potential.

References

[1] Cooper F, Khare A, Musto R and Wipf A 1988 *Ann. Phys.* **187** 1
[2] Clerk G J and McKellar B H J 1993 *Phys. Rev.* B **47** 6942
[3] Coutinho F A B and Nogami Y 1987 *Phys. Lett.* **124A** 211
[4] Sen Gupta N D 1989 *Phys. Lett.* **135A** 427
Percoco U and Villalba V M 1989 *Phys. Lett.* **141A** 221
Roy C L 1993 *Phys. Rev.* A **47** 3417
[5] Clerk G J and McKellar B H J 1992 *J. Phys. A: Math. Gen.* **25** L861
[6] Nogami Y and Toyama F M 1992 *Phys. Rev.* A **45** 5258
Toyama F M, Nogami Y and Zhao Z 1993 *Phys. Rev.* A **47** 897
Nogami Y and Toyama F M 1993 *Phys. Rev.* A **47** 1708
[7] Roy C L 1994 *Phys. Lett.* **189A** 345
[8] Thaller B 1992 *The Dirac Equation* (Berlin: Springer)
[9] Arkadiev V A, Pogrebkov A K and Polivanov M K 1983 *Sov. J. Theor. Math. Phys.* **53** 1051
[10] Hinton D B, Jordan A K, Klaus M and Shaw J K 1991 *J. Math. Phys.* **32** 3015
[11] Salle M A 1987 *Zap. Semin. Lomi* **161** 72
[12] Matveev V B and Sall' M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
[13] Sall' M A 1982 *Sov. J. Theor. Math. Phys.* **53** 1092
[14] Neugebauer G and Meinel R 1984 *Phys. Lett.* **100A** 467
[15] Chen Z-Y, Huang N-N, Lin Z-Z and Xiao Y 1993 *J. Phys. A: Math. Gen.* **26** 1365
[16] Beals R and Coifman R R 1987 *Inverse Problems* **3** 577
[17] Sabatier P C 1986 Reconstruction ambiguities of inverse scattering on the line *Proc. Conf. on Inverse Problems (Oberwolfach, 1986) (Internationale Schriftenreihe Numerische Mathematik 77)* ed J R Cannon and U Hornung (Basel: Birkhäuser) p 173

[18] Andersen A 1991 *Phys. Rev.* A **43** 4602
[19] Stahlhofen A A 1994 On supertransparent potentials of the Schrödinger equation *Phys. Rev.* A (accepted)
[20] von Neumann J and Wigner E 1929 *Z. Phys.* **30** 465
[21] Thompson M 1976 *Bollettino UMI* **13A** 576
Vogelsang W 1987 *Analysis* **7** 259
Mandache N 1994 *Lett. Math. Phys.* **31** 85
[22] Sappyanova L A 1982 *Vestnik Leningrad Univ. Ser. Math.* **10** 199 (Engl. transl.)
[23] Kaisher A I 1985 *Math. Notes* **37** 194 (Engl. transl.)
[24] Behncke H 1994 *J. Math. Phys.* **35** 1445
[25] Beutler R 1993 *J. Math. Phys.* **34** 3098
[26] Deift P A 1978 *Duke Math. J.* **45** 267
[27] Matveev V B 1994 *J. Math. Phys.* **35** 2995
[28] Stahlhofen A A 1992 *Ann. Phys., Lpz* **1** 554
[29] Beutler R, Stahlhofen A A and Matveev V B 1994 *Phys. Scr.* **50** 9
[30] Hernández E and Mondragón A 1994 *Phys. Lett.* **326B** 1
Mondragón A and Hernández E 1994 Physical aspects of the degeneracy of resonance *Preprint* IFUNAM FT 94-53 Inst. de Fiscica, UNAM Mexico (*Proc. Workshop on Interactions of Neutrons with Nuclei (Dubna, 1994)* (to appear))